

# A Search-free DOA Estimation Algorithm for Coprime Arrays

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## Abstract

In this paper, we propose a fast search-free method for direction-of-arrival (DOA) estimation with coprime arrays. A coprime array consists of two uniform linear subarrays with inter-element spacings  $M\lambda/2$  and  $N\lambda/2$ , where  $M$  and  $N$  are coprime integers and  $\lambda$  is the wavelength of the signal. Because uniform linear arrays enjoy many processing advantages over arbitrary geometry arrays, our strategy is to exploit the uniform linear structure of the subarrays in coprime arrays. We first estimate the DOA's for each uniform linear subarray separately and then combine the two estimates from the two subarrays. For combining the two estimates, we propose a method that projects the estimated point in the two-dimensional plane onto one-dimensional line segments that correspond to the entire angular domain. By doing so, we avoid the search step and consequently, we greatly reduce the computational complexity of the method.

## Index Terms

DOA estimation, coprime arrays, coprime sampling, uniform linear array

## I. INTRODUCTION

Direction-of-arrival (DOA) estimation using sensor arrays is a problem that is frequently encountered in many engineering areas including radar, sonar, and wireless communication, and it has been studied for several decades. For arbitrary geometry arrays, popular DOA estimation methods include beamspan [1], Capon [2], MUSIC [3] and several versions of maximum likelihood estimators [4]. For finding the optimal estimate, they all require a computationally expensive search step in a nonconvex space. For uniform linear arrays (ULAs), the array response vector has a Vandermonde form, which allows the

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search step to be replaced by polynomial rooting. These so called *search-free* algorithms include IQML [5], MODE (Method of Direction Estimation) [6], root-MUSIC [7] and ESPRIT [8]. Besides, there are also approaches that transform arbitrary arrays to equivalent ULAs so that one can take the advantage of efficient algorithms for ULAs, e.g., array interpolation [9], manifold separation [10] and Fourier-Domain (FD) root-MUSIC [11]. These approaches basically try to approximate the steering vectors by virtual ULAs. Their drawback is that for achieving satisfactory performance, they require a large number of virtual arrays, which entails increased complexity. Furthermore, they suffer fixed error at high signal-to-noise ratios (SNRs) and thus, do not converge to the Cramér-Rao bound as the SNR increases.

Coprime signal processing has recently been studied in [12], [13], [14]. It has been shown that coprime processing allows us to sample a signal sparsely and estimate the parameters of the signal at a higher resolution [13]. The parameters can be the frequencies of temporal signals or the DOA's of spatial signals. A coprime array consists of multiple ULAs with different inter-element spacings that are mutually coprime. Although the DOA estimation with ULAs has been thoroughly studied, there has not been much work for designing special algorithms that take advantage of the *partial* uniform linear structure of coprime arrays. In [12], the authors apply the MUSIC algorithm to coprime processing, where they mainly consider the problem from the perspective of degree of freedom, i.e., how many signals the coprime array can identify. To achieve additional degree of freedom with coprime arrays, that is, to enhance the rank of the covariance matrix of the received signal, they propose the use of spatial smoothing. Nevertheless, the proposed method is not search-free, and its accuracy and computational complexity have not been investigated in detail.

In this paper, we address the problem of DOA estimation with coprime arrays with the emphasis on estimation accuracy and computational complexity. We propose a search-free method that exploits the uniform linear structure of the subarrays of the coprime arrays. Considering that the DOA estimation using ULAs is fast and accurate, we first estimate the DOA for each subarray separately. Since the inter-element distance for each subarray is larger than half-wavelength of the signal, ambiguity is present and as a result, the DOA's cannot be uniquely determined. To resolve the ambiguity, we use a projection-like method that combines the estimates from different subarrays, where the correctness of the estimate is guaranteed as a consequence of the coprimeness.

The paper is organized as follows. In Section 2, we introduce the model and briefly review the DOA estimation with ULAs. The proposed algorithm is discussed in detail in Section 3 and the numerical experiments are presented in Section 4. Section 5 provides final remarks.

We use the following notation:  $(\cdot)^T$  and  $(\cdot)^H$  denote the transpose and the conjugate transpose,

respectively;  $\mathbb{C}$  refers to the complex space;  $\delta_{k,l}$  signifies the Kronecker delta;  $\mathbb{E}(\cdot)$  stands for the expectation operator; and  $\text{tr}\{\mathbf{A}\}$  is the trace of the matrix  $\mathbf{A}$ .

## II. DOA ESTIMATION FOR ULA

The problem of finding the DOA's of  $D$  narrowband plane waves impinging on a ULA of  $L$  sensors can be modeled as follows [15] :

$$\mathbf{y}(k) = \mathbf{Ax}(k) + \mathbf{w}(k), \quad k = 1, \dots, K, \quad (1)$$

where  $\mathbf{y}(k) \in \mathbb{C}^{L \times 1}$  is the signal received by the sensors at the  $k$ th time slot;  $K$  is the number of snapshots;  $\mathbf{x}(k) \in \mathbb{C}^{D \times 1}$  contains the complex envelopes of the emitter signals;  $\mathbf{w}(k)$  is the noise process; and

$$\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_D] \in \mathbb{C}^{L \times D} \quad (2)$$

with

$$\begin{aligned} \mathbf{a}_d &= [1, \exp(jp_1\psi_d), \dots, \exp(jp_{L-1}\psi_d)]^\top \in \mathbb{C}^{L \times 1} \\ d &= 1, \dots, D; \end{aligned} \quad (3)$$

$p_l\lambda/2$  is the distance from the  $l$ th sensor to the reference point;  $\psi_d$  is the DOA of our interest. The number of sources,  $D$ , is assumed to be known. Thus,  $\Psi = [\psi_1, \dots, \psi_D]$  is the vector of DOA's we wish to estimate. We assume the signal  $\mathbf{x}(k)$  and the noise  $\mathbf{w}(k)$  are independent zero-mean complex Gaussian random processes with the following moments:

$$\mathbb{E}[\mathbf{x}(k)\mathbf{x}^H(l)] = \mathbf{P}\delta_{k,l}, \quad \mathbb{E}[\mathbf{x}(k)\mathbf{x}^T(l)] = \mathbf{0}, \quad k \neq l \quad (4)$$

$$\mathbb{E}[\mathbf{w}(k)\mathbf{w}^H(l)] = \sigma^2 \mathbf{I}\delta_{k,l}, \quad \mathbb{E}[\mathbf{w}(k)\mathbf{w}^T(l)] = \mathbf{0}, \quad k \neq l \quad (5)$$

with  $\mathbf{I}$  being the identity matrix,  $\mathbf{P}$ , the signal covariance matrix, and  $\sigma^2$ , the noise power. The covariance matrix of the received signal  $\mathbf{y}(k)$  can be written as

$$\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}. \quad (6)$$

We denote by  $\hat{\mathbf{R}}$  the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{y}(k)\mathbf{y}^H(k). \quad (7)$$

The eigendecomposition of  $\hat{\mathbf{R}}$  in (7) can be written as

$$\hat{\mathbf{R}} = \hat{\mathbf{E}}_s \hat{\Lambda}_s \hat{\mathbf{E}}_s^H + \hat{\mathbf{E}}_n \hat{\Lambda}_n \hat{\mathbf{E}}_n^H, \quad (8)$$

where  $\hat{\Lambda}_s \in \mathbb{C}^{D \times D}$  and  $\hat{\Lambda}_n \in \mathbb{C}^{(L-D) \times (L-D)}$  are diagonal matrices that contain the eigenvalues of the signal and the noise subspaces, respectively, whereas  $\hat{\mathbf{E}}_s \in \mathbb{C}^{L \times D}$  and  $\hat{\mathbf{E}}_n \in \mathbb{C}^{L \times (L-D)}$  are composed of the eigenvectors of the signal and the noise subspaces, respectively.

For ULAs, the steering vector becomes

$$\mathbf{a}_i(\psi_i) = [1, \exp(j\psi_i), \dots, \exp(j(L-1)\psi_i)]^\top. \quad (9)$$

There exist a large number of fast algorithms for estimating DOA's with ULAs, among which MODE is the leading candidate [16], [17]. Since we later use MODE as part of our algorithm, we review it here briefly. The MODE algorithm was originally proposed in [6]. It estimates the DOA's via the minimization of the following function:

$$\mathbf{b} = \arg \min_{\mathbf{b} \in \mathbb{C}^{D \times 1}} \text{tr}\{\Pi_{\mathbf{B}} \hat{\mathbf{E}}_s \mathbf{W} \hat{\mathbf{E}}_s^H\} \quad (10)$$

where

$$\mathbf{b} = [b_0, b_1, \dots, b_D]^\top, \quad (11)$$

$$\Pi_{\mathbf{B}} = \mathbf{B}(\mathbf{B}^H \mathbf{B})^{-1} \mathbf{B}^H, \quad (12)$$

$$\mathbf{W} = (\hat{\Lambda}_s - \hat{\sigma}^2 \mathbf{I})^2 \hat{\Lambda}_s^{-1}, \quad (13)$$

$$\hat{\sigma}^2 = \frac{1}{L-D} \text{tr}\{\hat{\Lambda}_N\}, \quad (14)$$

and  $\mathbf{B}$  is a  $L \times (L-D)$  Toeplitz matrix defined by

$$\mathbf{B}^H = \begin{bmatrix} b_D & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_D & \cdots & b_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & b_D & \cdots & b_0 \end{bmatrix}. \quad (15)$$

Let  $z = \exp(j\psi_i)$ , and denote by  $b(z)$  the complex-valued polynomial

$$b(z) = b_0 z^D + b_1 z^{D-1} + \cdots + b_D \quad (16)$$

$$= b_0 \prod_{i=1}^D (z - z_i). \quad (17)$$

The angles of the roots of the polynomial are the estimates of the DOA's. It was shown in [6] that the MODE algorithm is an asymptotically efficient estimator of the DOA's. To minimize (10), both iterative and noniterative steps have been proposed. See [16], [17] for detail.

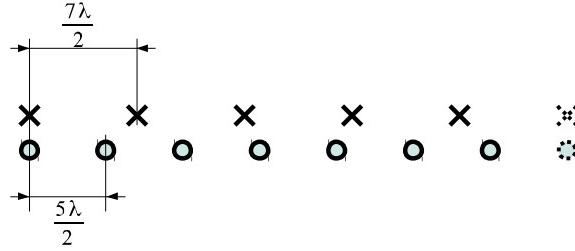


Fig. 1. a coprime array with two ULAs, where  $M = 7$  and  $N = 5$ .

### III. DOA ESTIMATION FOR COPRIME ARRAYS

We consider a coprime array with two uniform linear subarrays. We assume that the two subarrays have inter-element spacing  $M\lambda/2$  and  $N\lambda/2$ , respectively with  $M$  and  $N$  being coprime. Fig. 1 shows the case for  $N = 7$  and  $M = 5$ . Because the two subarrays share the first sensor, the total number of sensors  $L$  is equal to  $M + N - 1$ . The corresponding steering vector in (3) becomes

$$\mathbf{a}_d = \left[ 1, e^{j\psi_d M}, e^{j\psi_d 2M}, \dots, e^{j\psi_d (N-1)M} \right. \\ \left. e^{j\psi_d N}, e^{j\psi_d 2N}, \dots, e^{j\psi_d (M-1)N} \right]^\top. \quad (18)$$

For such arrays, the efficient methods for ULAs are not directly applicable. We can of course use the algorithms for arbitrary arrays, but they are slow and the uniform linear structure of the subarrays is wasted. Our objective is to develop an algorithm that is applicable to coprime arrays while at the same time enjoys the efficiency brought by the uniform linear structure of the subarrays. The proposed approach is composed of two steps. We first estimate the DOA's from each subarray separately and then combine the two estimates in an innovative way.

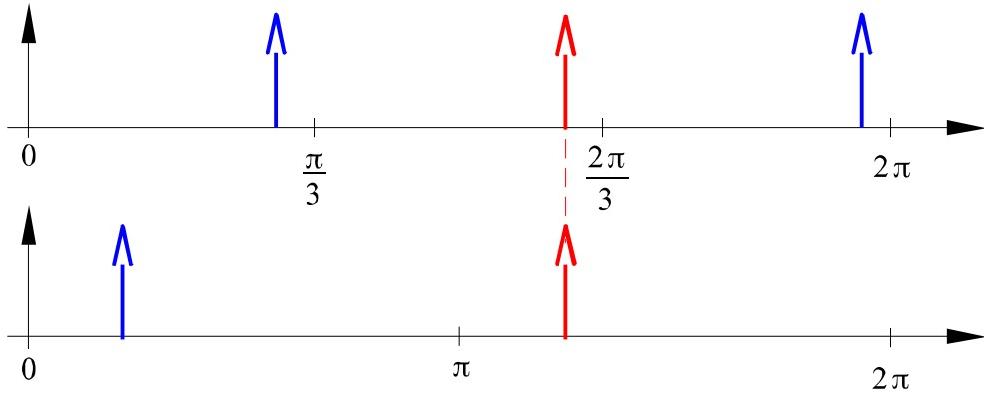


Fig. 2. Ambiguity on the angular domain, case 1.

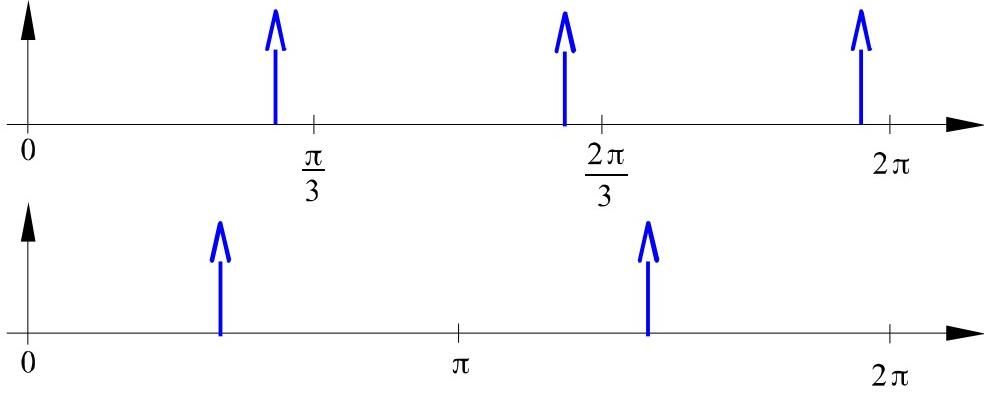


Fig. 3. Ambiguity on the angular domain, case 2.

#### A. DOA estimation using a single subarray

For a subarray, the inter-element spacing is  $N\lambda/2$  with  $N > 1$ . This has the same effect as we undersample a signal by a factor of  $N$  in the temporal domain. According to the basic sampling theorem, we will have ambiguities, or aliasing, in the angular domain. Specifically, if we use the same algorithm as described above, the polynomial in (17) becomes

$$b'(z) = b'_0 \prod_{i=1}^D (z^N - z'_i). \quad (19)$$

This polynomial has  $N \times D$  roots instead of  $D$ . If  $z_i$  is a root of (17), it is still be a root of (19). The problem is that aliasing occurs with the period of  $2\pi/N$ ;  $z_i \exp(jn\frac{2\pi}{N})$  for  $n = 1, \dots, N-1$  are also the roots of (19), as shown in Fig. 2. Therefore, we have no way of distinguishing the  $z_i$  we want from the remaining  $N-1$  roots.

Note that in this step, it does not matter which algorithm is used. In the experiment, we use MODE, which was shown to have the best performance for ULAs [17].

#### B. Combination of the estimates from subarrays

The present ambiguities in the estimates from the two subarrays notwithstanding, we show that we are able to disambiguate them. For example, in the case of one source and no noise, in Fig. 2, we can see that the estimates from the two subarrays coincide at the red arrows. Therefore, we can be very confident that the position of the red arrow is the DOA. In the presence of noise, however, coincidences as the one in Fig. 3 are unlikely, and consequently, it becomes difficult to tell the exact DOA. Furthermore, in the case with multiple sources, obtaining the DOA's in this way becomes impossible.

We propose a projection method that can uniquely determine the exact DOA. Recall that the aliasing period in the angular domain for the subarray is  $2\pi/N$ . Thus, we only need to take the values between  $[-\pi, -\pi+2\pi/N]$ . The ranges of the outputs of the two subarrays define a rectangle in the two-dimensional plane as shown in Fig. 4, where  $N = 2$  and  $M = 3$ . It is not difficult to see that the 45 degree oblique line segments colored in blue correspond to the entire angular domain. The Chinese remainder theorem guarantees that the map is one-to-one and onto as a result of the coprimeness. This is the key point of our algorithm. In Fig. 4, the entire angular domain consist of four line segments; L1 corresponds to  $[-\pi, -\pi/3]$ ; L2,  $[-\pi/3, 0]$ ; L3,  $[0, \pi/3]$ ; and L4,  $[\pi/3, \pi]$ . For general  $M$  and  $N$ , there will be  $M + N - 1$  oblique line segments. Suppose the outputs of the first and the second subarray are  $\psi^{(1)}$  and  $\psi^{(2)}$ , respectively. The two estimates specify a point in the plane. We project the point onto the oblique line segments that corresponds to the entire angular domain. Specifically, we seek the point on the oblique line segments that is nearest to that point specified by the two estimates. This ensures that the combination is optimal and the result is valid. For point B in Fig. 4, we can simply draw a line that is vertical to the oblique line and measure the distance from B to the intersections. For point A, however, one intersection falls outside the rectangle area and should be wrapped around. As a result, the intersection is actually on L3 in the figure. After the nearest point on the line segments is found, it is easy to calculate the corresponding DOA by solving a set of modular linear equations, or it can be precalculated and made into lookup table.

### C. Combination of the estimates in the case of multiple sources

In the case of multiple sources, the method gets more complicated. Suppose that the outputs of the first subarray are  $\psi_1^{(1)}$  and  $\psi_2^{(1)}$  and the outputs of the second subarray,  $\psi_1^{(2)}$  and  $\psi_2^{(2)}$ . We have no way of knowing which value corresponds to the first and which to the second target. Therefore, we have to check two possibilities. In Fig. 4,  $\psi_1^{(1)}$ ,  $\psi_2^{(1)}$ ,  $\psi_1^{(2)}$  and  $\psi_2^{(2)}$  define four points on the two-dimensional plane, which are marked  $A$ ,  $B$ ,  $C$ , and  $D$ . Accordingly, we can choose for projection the estimates given by the pair  $(A, B)$  or  $(C, D)$ . To decide which pair is more likely, we can simply choose the one with the minimum sum of distances to their nearest oblique lines. For the case of  $D$  targets the number of possible combinations is equal to the factorial of  $D$ . Although this may be very large for large  $D$ , we can use greedy algorithm to find the optimal combination with linear time complexity.

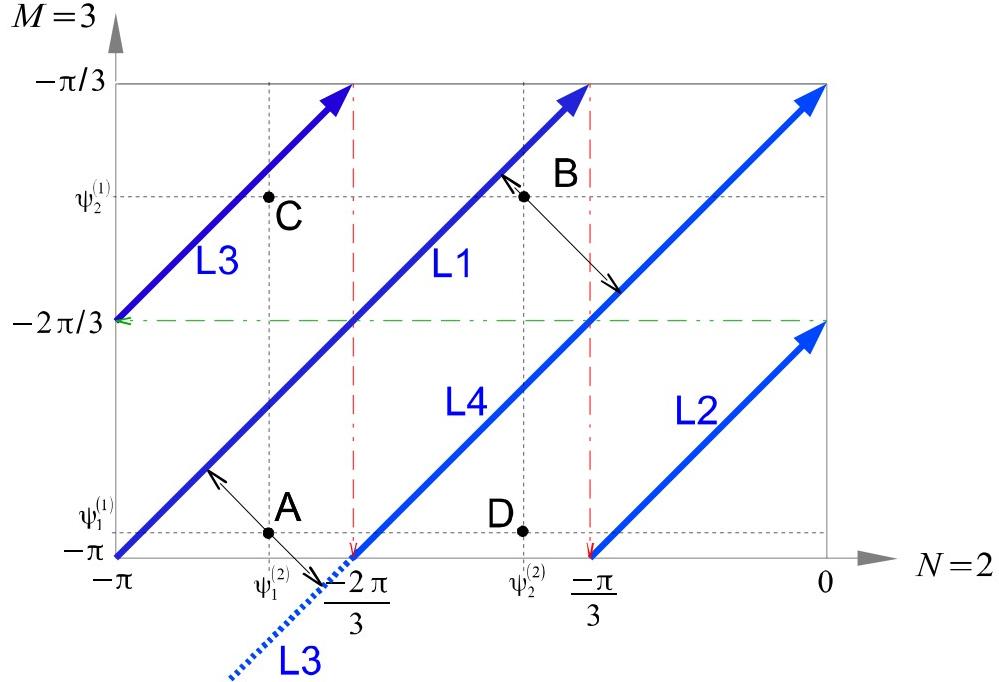


Fig. 4. Projecting the point specified by the two estimates onto the entire angular domain.

#### IV. SIMULATION

In this section, we use a numerical experiment to justify our proposed method. We show the performance comparison between the proposed method, the FD root-MUSIC method, and the Cramér-Rao bound. The FD root-MUSIC uses Fourier series coefficients to approximate the null-spectrum function [11]. It was shown to have better performance than other search-free methods [18].

In the simulations, we set  $N = 7, M = 5$ . For the FD root-MUSIC, 99 Fourier series coefficients are used for approximation. We only test the case of one source with  $\psi_1 = 0.1\pi$ . Fig. 5 shows the mean square error (MSE) performance for different values of SNR. We can see that at low SNRs, the FD root-MUSIC outperforms the proposed method. At high SNRs, however, the performance of the proposed method approaches the CRB while the FD root-MUSIC suffers fixed error due to its approximation. In Fig. 6, we plotted the MSE performance for different number of snapshots. Similarly, the FD root-MUSIC is better when the number of snapshots is small. On the other hand, the proposed method approaches the CRB when  $K > 600$  while the FD root-MUSIC does not improve when  $K > 300$ .

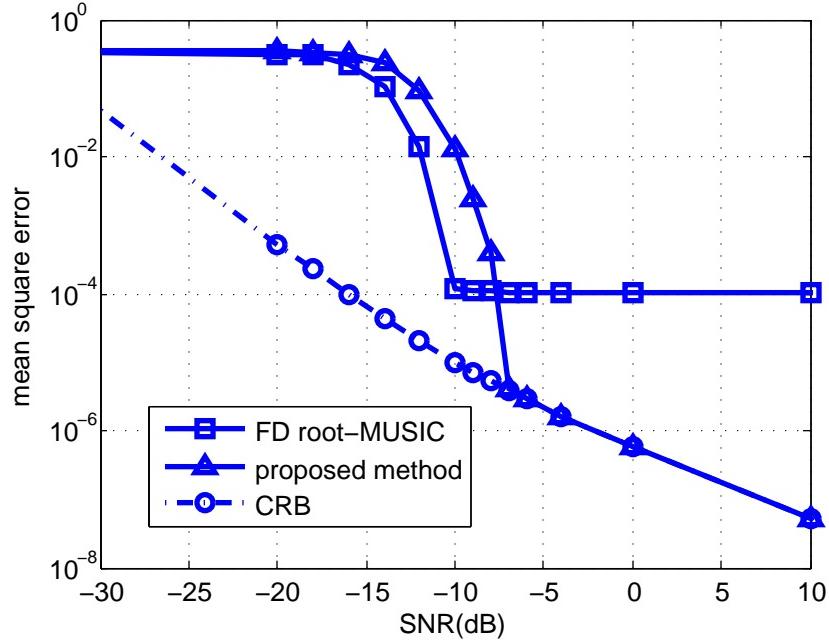


Fig. 5. Mean square error performance at different SNRs. ( $K=100$ )

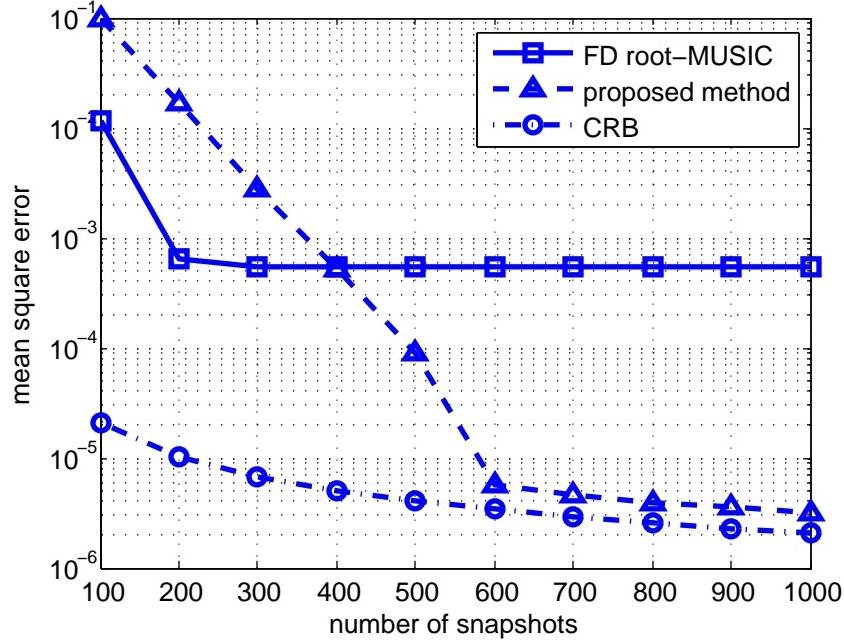


Fig. 6. Mean square error performance for different number of snapshots. (SNR=-12dB)

## V. CONCLUDING REMARKS

In this paper, a fast search-free DOA estimation algorithm for coprime arrays is proposed. It exploits the uniform linear structure of the subarrays by first estimating the DOA separately within each subarray and then combining the estimates from different subarrays. Simulation shows that the performance of the

proposed algorithm is close to the FD root-MUSIC method at low SNRs and much better at high SNRs. We point out that the proposed method has much lower complexity.

In the combination step, we seek the point with the shortest distance under an implicit assumption that the error variances along the two dimensions are equal. In practice, however, this may not be the case. To take the different error variances into consideration, one needs to construct an oblique projection to find the point on the line segments.

Here, we only discussed the case with two subarrays. For a coprime array with  $m > 2$  subarrays, the state space will be  $m$  dimensional hyperrectangle, but the entire angular domain can still be mapped to line segments and the same idea for estimation can be reproduced. Therefore, the presented results can be extended to coprime arrays with multiple subarrays.

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